*J. Fluid Mech.* (1967), *vol.* 28, *part* 3, *pp.* 571–592
*Printed in Great Britain*



# On the suppression of turbulence by a uniform magnetic field

By H. K. MOFFATT

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge

(Received 28 September 1966)

The suppression of initially isotropic turbulence by the sudden application of a uniform magnetic field is considered. The problem is characterized by three dimensionless numbers, a Reynolds number $R$, a magnetic Reynolds number $R_m$ and a magnetic interaction parameter $N$ (for definitions, see equations (1.2) and (1.4)). It is supposed that $R \gg 1$, $R_m \ll 1$ and $N \gg 1$. There are two important time scales, $t_d$ a time characteristic of magnetic suppression, and $t_0 = Nt_d$, the 'turn-over' time of the turbulent energy-containing eddies. For $0 < t \ll t_0$ the response of the energy-containing components of the turbulence to the applied field is linear and the time dependence of the kinetic energy density $K(t)$ and magnetic energy density $M(t)$ are analysed. There are essentially two distinct contributions to each from two domains of wave-number space $\mathscr{D}_1$ and $\mathscr{D}_2$ (defined in figure 2). In $\mathscr{D}_1$ the response is severely anisotropic, while in $\mathscr{D}_2$ it is nearly isotropic. The relative importance of the contributions $K_1(t)$ (from $\mathscr{D}_1$) and $K_2(t)$ (from $\mathscr{D}_2$) to $K(t)$ depends on the value of the Lundquist number $S = (NR_m)^{\frac{1}{2}}$. If $S \ll 1$, then $K_1(t)$ dominates for all $t \lesssim t_0$ and $K(t) \propto t^{-\frac{1}{2}}$ for $t_d \ll t \ll t_0$. If $S \gg R_m^{-2}$, then $K_2(t)$ dominates, and $K(t) \propto t^{-\frac{3}{2}}$ for $R_m t_0 \ll t \ll t_0$. If $1 \ll S \ll R_m^{-2}$, then a changeover in the dominant contribution occurs when $t = O(S^{\frac{1}{2}} R_m) t_0$. Analogous results are obtained for the magnetic energy density.

## 1. Introduction

It has been pointed out by Lehnert (1955) that a uniform applied magnetic field $\mathbf{B}_0$ has a pronounced effect on the decay of turbulence in a conducting fluid. If the kinematic viscosity of the fluid $\nu$ is small compared with its magnetic diffusivity $\lambda$, and if circumstances are such that non-linear terms in the governing equations are negligible, then the effect of the field is to suppress preferentially those Fourier components of the velocity field whose wave vector $\mathbf{k}$ has a non-zero component parallel to $\mathbf{B}_0$, i.e. which tend to bend the field lines. Fourier components for which $\mathbf{k}.\mathbf{B}_0 = 0$ are unaffected by the field—an effect which is also well known and of profound importance in the related context of stability theory. The time characteristic of this 'preferential damping' process is

$$t_d = \lambda / h_0^2, \tag{1.1}$$

where $\mathbf{h}_0 = (\mu\rho)^{-\frac{1}{2}}\mathbf{B}_0$, $\mu$ is the magnetic permeability and $\rho$ the density (assumed uniform) of the fluid.

A second effect of the magnetic field is that a certain range of Fourier components of the velocity field of small wave-number tend to propagate as Alfvén waves. If the dominant contribution to the energy of the turbulence comes from such components, then the turbulence will have the character of a random sea of interacting Alfvén waves. In particular, during the last stages of decay of homogeneous turbulence in the presence of a uniform field, when the length scale of the energy-containing eddies increases without limit, the turbulence must have this character. The final period of decay has been the subject of several papers, following Lehnert (1955); the relation of these contributions to the present study will be established in the appendix.

The specific situation studied in this paper is the following. Suppose that for $t < 0$ a conducting fluid is in a state of turbulent motion characterized by a root-mean-square velocity $u_0$, and a length scale (of energy-containing eddies) $l_0$, with

$$R \equiv \frac{u_0 l_0}{\nu} \gg 1, \quad R_m \equiv \frac{u_0 l_0}{\lambda} \ll 1. \tag{1.2}$$

The time scale of the energy-containing eddies (the 'turnover' time) is

$$t_0 = l_0/u_0. \tag{1.3}$$

Suppose that, at the instant $t = 0$, a uniform magnetic field $\mathbf{B}_0$ is 'switched on' (whether this is a physical possibility is a separate matter which will be discussed below), and suppose that

$$N \equiv \frac{t_0}{t_d} = \frac{h_0^2 l_0}{\lambda u_0} \gg 1, \tag{1.4}$$

i.e. $t_d \ll t_0$. $N$ is usually described as the magnetic interaction parameter and may also be interpreted (when $R_m \ll 1$) as the ratio of the magnitude of the Lorentz force $|(\nabla \wedge \mathbf{h}) \wedge \mathbf{h}_0|$ to the magnitude of the non-linear part of the inertia force $|\mathbf{u}.\nabla\mathbf{u}|$ in the equation of motion.† The preferential damping of the turbulence then proceeds on a time scale short compared with the time $t_0$ characteristic of the 'energy transfer' associated with the non-linear forces. For $0 < t \ll t_0$, the turbulence rapidly adjusts to the new externally applied conditions, and during this period of adjustment‡ the non-linear forces are negligible (as are viscous forces *a fortiori*), and the linear decay equations studied by Lehnert are applicable. For $t = O(t_0)$ and greater, non-linear forces may again become important in determining the new spectral distribution of energy.

It is well known in other contexts (Batchelor 1953, chap. IV) that, if external conditions are suddenly changed, then turbulence will respond in a linear manner

† If $N \ll 1$, then the Lorentz force is small compared with the inertia force, and may be expected to have negligible effect on the turbulence. In this case, the turbulence generates weak electric currents whose spectrum is related in a simple way to the spectrum of the velocity field (see Liepmann 1952; Golitsyn 1960; or the review article by Moffatt 1962).

‡ This statement requires refinement when the details of the non-linear process are analysed. It is possible that non-linear effects become important after a time of order $t_n = N^{\gamma} t_d$ where $0 < \gamma < 1$. The value of $\gamma$ depends on the detailed nature of the non-linear process, which has not yet been satisfactorily analysed. In any case, it is certain that $t_n \gg t_d$ if $N$ is sufficiently large and linear theory is valid until $t = O(t_n)$. We shall use the preliminary crude estimate $\gamma = 1$ until analysis definitely reveals an inconsistency.

during a time small compared with the time $t_0$. The damping of turbulence by convection through a wire gauze, and the distortion of turbulence by passage through a duct of rapidly varying cross-section have been analysed by linear techniques. In both cases the theory attempts only to relate the statistical properties of the turbulence immediately after the change of conditions (i.e. for $t \ll t_0$) to its properties immediately before.

The feasibility of switching on a field $\mathbf{B}_0$ at the instant $t = 0$ requires some comment. If we imagine that the turbulent fluid is contained in a cubical box of side $L_0 \gg l_0$, and if the source of the field is an electromagnet with coils surrounding the box, then it takes a finite time of order $t_s = \lambda^{-1}L_0^2$ from the moment of applying power to the electromagnet, for the field to penetrate to the centre of the box; $t_s$ may be described as the switch-on time for the field, and, if the process

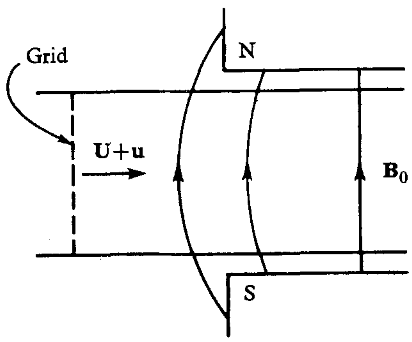



FIGURE 1. The sudden application of a magnetic field; grid turbulence is convected into the region of uniform field between magnet poles N, S.

is to be regarded as instantaneous, $t_s$ must be small compared with $t_d$, and clearly this condition can be satisfied only if

$$S \equiv h_0 l_0/\lambda = (NR_m)^{\frac{1}{2}} \ll l_0/L_0. \tag{1.5}$$

$S$ is the Lundquist number, and it will play a central role in the analysis of this paper. This suggests that, for strictly homogeneous turbulence ($L_0 \to \infty$), the 'switch-on' specification of the problem is legitimate only in the limit $S \to 0$; alternatively, one could say that the switching on of the field will introduce inhomogeneities on the scale $S^{-1}l_0$; provided $S \ll 1$, it is legitimate to assume that homogeneity persists, at any rate on the scale $l_0$ of the energy-containing eddies.

If $S \ll 1$, the simplest way to realize the above conditions in the laboratory might be simply to sweep grid turbulence into a region of uniform field (figure 1). The switch-on time is then $t_s = d/U$, where $U$ is the uniform mean velocity and $d$ the fringeing distance of the field, i.e. approximately the gap width of the magnet N–S. The conditions $t_s \ll t_d \ll t_0$ are then satisfied provided

$$\frac{U}{d} \gg \frac{h_0^2}{\lambda} \gg \frac{u_0}{l_0},$$

and, since the ratio $u_0/U$ can be made small simply by decreasing the dynamical resistance of the grid used, it should be possible to satisfy these conditions. (Of course, the magnetic Reynolds number based on $U$, $Ud/\lambda$, must remain small, since, otherwise, the field of the magnet would be significantly distorted.)

If $S \gg 1$, then strong inhomogeneity must develop if a field is switched on as described above. For turbulence in a box, one would expect the eddies near the boundaries to be first affected by the inward-diffusing field. The problems raised are certainly interesting, but mathematically intractable. An alternative specification of the problem which does not exclude homogeneity is as follows. Suppose that, for $t < 0$, the fluid is at rest and permeated by a uniform field $\mathbf{B}_0$, and suppose that, at the instant $t = 0$, a homogeneous random velocity field is generated by random impulsive forces (Saffman 1967). In principle, any initial velocity field $\mathbf{u}(\mathbf{x}, 0)$ may be generated by an impulsive force distribution $\rho^{-1}\mathbf{u}(\mathbf{x}, 0)\,\delta(t)$; in the absence of any 'impulsive electromotive forces', magnetic field perturbations $\mathbf{h}(\mathbf{x}, t)$ take a finite time to develop, so that $\mathbf{h}(\mathbf{x}, 0) = 0$. The problem then is to investigate how $\mathbf{u}(\mathbf{x}, t)$ evolves statistically, particularly during the 'initial phase' $0 < t \ll t_0$. It is by no means obvious whether such conditions can be realized, even approximately, under laboratory conditions. Nevertheless, the problem is of considerable fundamental interest; it is essential to understand fully how a field of turbulence is initially modified by a uniform magnetic field, before there can be any hope of understanding fully non-linear effects.

When $R_m \ll 1$, the induced field $\mathbf{h}(\mathbf{x}, t)$ is of order $R_m h_0$ and satisfies the linearized induction equation (Lehnert 1955)

$$\partial\mathbf{h}/\partial t = \mathbf{h}_0 . \nabla\mathbf{u} + \lambda\nabla^2\mathbf{h}. \tag{1.6}$$

The relevant initial condition for the problem studied here is $\mathbf{h}(\mathbf{x}, 0) \equiv 0$. The velocity field $\mathbf{u}(\mathbf{x}, t)$ is determined by the equation of motion, and for $0 < t \ll t_0$ as explained above (assuming $N \gg 1$) it takes the linearized inviscid form

$$\partial\mathbf{u}/\partial t = -\rho^{-1}\nabla P + \mathbf{h}_0 . \nabla\mathbf{h}, \tag{1.7}$$

where $P$ is the sum of the fluid pressure and the magnetic pressure. The initial conditions here are that all statistical properties of the field $\mathbf{u}(\mathbf{x}, 0)$ are supposed given. In order to carry out the limited analysis of this paper, only the spectrum tensor of the field $\mathbf{u}(\mathbf{x}, 0)$ will be required; a discussion of the most appropriate initial conditions will be deferred to §3. The fields $\mathbf{u}$ and $\mathbf{h}$ also satisfy the kinematical conditions

$$\nabla . \mathbf{u} = \nabla . \mathbf{h} = 0. \tag{1.8}$$

One of the aims of this paper will be to determine the time dependence of such quantities as the kinetic energy density during such time intervals as that defined by $t_d \ll t \ll t_0$. Although the physical meaning of such an interval may be clear, its mathematical definition may require a word of comment. Two limiting processes are considered, $t_0/t_d = N \to \infty$ and $t \to \infty$. If the two processes are 'tied', in such a way that $t = o(N)$, e.g. $t = O(N^{\frac{1}{2}})$, then the two limits

$$t/t_d \to \infty, \quad t/t_0 \to 0, \tag{1.9}$$

are compatible. The notation

$$\psi(t, N) \sim f(t) \quad (t_d \ll t \ll Nt_d)$$

will be used to mean the same as

$$\lim_{\substack{t\to\infty,\, N\to\infty \\ t=o(N)}} \frac{\psi(t,N)}{f(t)} = 1.$$

Similarly, for example, the notation

$$\psi(t,N) = O(t^{-3}) \quad (t_d \ll t \ll Nt_d)$$

will mean

$$\lim_{\substack{t\to\infty,\, N\to\infty \\ t=o(N)}} t^3\psi(t,N) = \text{const.}$$

In order to avoid tedious repetition, we shall often simply write $t \gg t_d$ and omit the explicit dependence on $N$, but it must be stressed that the analysis of § 3 is liable to break down when $t = O(Nt_d)$.

Some of the statistical properties of the solutions of Lehnert's equations have been numerically computed by Deissler (1963). Those of his results that are relevant to the present study will be discussed in § 3.

A preliminary calculation of the initial response of turbulence to the application of a strong magnetic field, by means of a Taylor series in time, has been carried out by Nestlerode & Lumley (1963). This provides an indication of the response while $t \ll t_d$, when the suppression effect is still very slight. The present paper goes very much further, by clarifying the nature of the suppression process when $t \gg t_d$ and the turbulence has been very considerably modified.

## 2. The suppression of a single Fourier component

Equations (1.6) and (1.7) are well known as the equations that describe decaying Alfvén waves. Certain properties of the solutions of the equations that are of central importance in later sections are recapitulated here. Let us restrict attention to fields which admit a Fourier representation of the form

$$\left.\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \int \mathbf{p}(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x})\,d\mathbf{k}, \\ \mathbf{h}(\mathbf{x},t) &= \int \mathbf{q}(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x})\,d\mathbf{k}, \\ P(\mathbf{x},t) &= \int \Pi(\mathbf{k},t)\exp(i\mathbf{k}.\mathbf{x})\,d\mathbf{k}, \end{aligned}\right\} \tag{2.1}$$

where $\mathbf{k} = (k_1, k_2, k_3)$, $d\mathbf{k} = dk_1 dk_2 dk_3$, and the $k_3$-axis is chosen parallel to $\mathbf{h}_0$. Then the Fourier transforms of equations (1.6), (1.7) and (1.8) are

$$\left.\begin{aligned} \partial\mathbf{p}/\partial t &= -i\mathbf{k}\Pi/\rho + i(\mathbf{h}_0.\mathbf{k})\,\mathbf{q}, \\ [(\partial/\partial t) + \lambda k^2]\,\mathbf{q} &= i(\mathbf{h}_0.\mathbf{k})\,\mathbf{p}, \\ \mathbf{k}.\mathbf{p} = \mathbf{k}&.\mathbf{q} = 0. \end{aligned}\right\} \tag{2.2}$$

It follows immediately that $k^2\Pi = 0$, and hence that $\mathbf{k}\Pi \equiv 0$. Equations (2.2) then admit decaying solutions, $\mathbf{p}, \mathbf{q} \propto e^{-\beta t}$, where $\beta$ is a root of the quadratic equation

$$-\beta(-\beta + \lambda k^2) + (\mathbf{h}_0.\mathbf{k})^2 = 0,$$

i.e.

$$\beta = \tfrac{1}{2}\lambda k^2[1 \pm (1-\zeta^2)^{\frac{1}{2}}] = \beta_1, \beta_2, \quad \text{say;} \tag{2.3}$$

here $\zeta$ is the parameter introduced by Lehnert (1955)

$$\zeta = 2\mathbf{h}_0 . \mathbf{k}/k^2\lambda = (2k_d/k)\cos\theta, \tag{2.4}$$

where $k_d$, the 'dissipation wave-number', is defined by

$$k_d = h_0/\lambda, \tag{2.5}$$

and $\theta$ is the angle between $\mathbf{h}_0$ and $\mathbf{k}$.

As observed by Lehnert, the character of the decay depends critically on the value of $\zeta$; if $|\zeta| < 1$ both roots $\beta_1$ and $\beta_2$ are real and positive, and $\mathbf{p}$ and $\mathbf{q}$ decay exponentially without oscillation, while, if $|\zeta| > 1$, the roots are complex conjugates and damped oscillating solutions occur. It will be expedient to consider

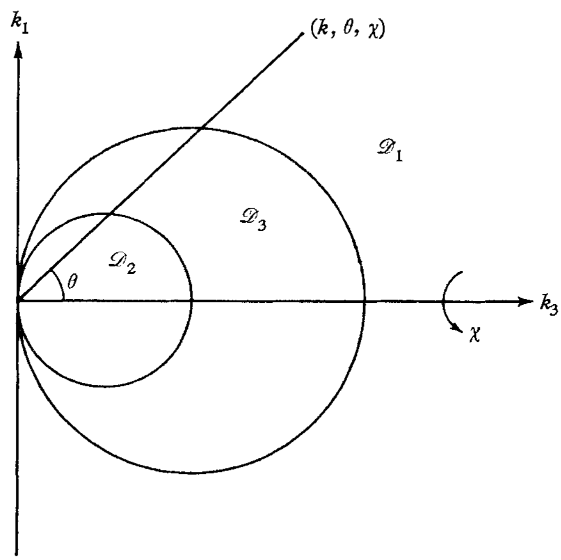



FIGURE 2. The domains $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_3$ of wave-number space. Only the half-space $\cos\theta \geqslant 0$ is indicated. The complete picture is symmetrical about the axis $\theta = 0$ and about the plane $\theta = \frac{1}{2}\pi$. The locus $\zeta = \delta$ is the sphere of radius $2\delta k_d$ touching the $k_1k_2$-plane at $\mathbf{k} = 0$.

separately the behaviour of the solutions in the following three domains (figure 2) of wave-number space

$$\mathscr{D}_1\text{: } |\zeta| < \delta; \quad \mathscr{D}_2\text{: } |\zeta| > \delta^{-1}; \quad \mathscr{D}_3\text{: } \delta < |\zeta| < \delta^{-1}, \tag{2.6}$$

where $\delta$ is any fixed number, $0 < \delta < 1$; to be specific, and for numerical convenience, let $\delta = 2^{-\frac{1}{2}}$. At a later stage (see (3.41) below), we shall let $\delta \to 1$, so that $\mathscr{D}_3$ will disappear.

### (i) *The decay of Fourier components in* $\mathscr{D}_1$

Here define a new variable $\xi$ by

$$\sin\xi = \zeta, \quad |\xi| < \tfrac{1}{4}\pi. \tag{2.7}$$

Then, from (2.3),

$$\beta_1 = \lambda k^2 \cos^2 \tfrac{1}{2}\xi, \quad \beta_2 = \lambda k^2 \sin^2 \tfrac{1}{2}\xi, \tag{2.8}$$

and it is readily verified that the solution of (2.2) subject to the initial conditions

$$\mathbf{p}(\mathbf{k},0) = \mathbf{p}_0(\mathbf{k}), \quad \mathbf{q}(\mathbf{k},0) = 0, \tag{2.9}$$

is

$$\left.\begin{aligned} \mathbf{p} &= \tfrac{1}{2}\tan\xi(e^{-\beta_2 t}\cot\tfrac{1}{2}\xi - e^{-\beta_1 t}\tan\tfrac{1}{2}\xi)\,\mathbf{p}_0, \\ \mathbf{q} &= \tfrac{1}{2}i\tan\xi(e^{-\beta_2 t} - e^{-\beta_1 t})\,\mathbf{p}_0. \end{aligned}\right\} \tag{2.10}$$

For $|\zeta| \ll 1$, $\xi \approx \zeta$ and these results take the simpler approximate form

$$\beta_1 \approx \lambda k^2, \quad \beta_2 \approx \lambda k_d^2 \cos^2\theta, \tag{2.11}$$

and

$$\mathbf{p} \approx \mathbf{p}_0 e^{-\beta_2 t}, \tag{2.12}$$

$$\mathbf{q} \approx \tfrac{1}{2}i\zeta(e^{-\beta_2 t} - e^{-\beta_1 t})\,\mathbf{p}_0 \sim \tfrac{1}{2}i\zeta\mathbf{p}_0 e^{-\beta_2 t} \quad (t \gg (\lambda k^2)^{-1}). \tag{2.13}$$

The magnetic energy in such a Fourier component is a factor $\frac{1}{4}\zeta^2$ smaller than the kinetic energy.

### (ii) *The decay of Fourier components in $\mathscr{D}_2$*

Here the substitution $\zeta = \cosh\xi$ is appropriate, and the solution of equation (2.2) subject to the initial conditions (2.9) is

$$\left.\begin{aligned} \mathbf{p} &= \mathbf{p}_0\left(\cos\omega t + \frac{\sin\omega t}{\sinh\xi}\right)e^{-\frac{1}{2}\lambda k^2 t}, \\ \mathbf{q} &= i\mathbf{p}_0\coth\xi\sin\omega t\, e^{-\frac{1}{2}\lambda k^2 t}, \end{aligned}\right\} \tag{2.14}$$

where $\omega = \frac{1}{2}\lambda k^2 \sinh\xi$. When combined with the factor $e^{i\mathbf{k}\cdot\mathbf{x}}$, this represents a damped 'standing' Alfvén wave of well-known type. When $|\zeta| \gg 1$, $|\coth\xi| \approx 1$, and the energy of the Fourier component (2.14) is equally divided between the velocity and the magnetic fields.

### (iii) *The decay of Fourier components in $\mathscr{D}_3$*

Although the solution (2.10) will be used only in $\mathscr{D}_1$, it is clearly valid in the wider domain $|\zeta| < 1$. Likewise the solution (2.14) is valid in $|\zeta| > 1$. Both solutions behave in a singular manner as $|\zeta| \to 1$; hence the need to consider separately the domain $\mathscr{D}_3$.

For $|\zeta| = 1$, $\beta_1 = \beta_2 = \frac{1}{2}\lambda k^2$, and the solution of (2.2) subject to (2.9) is

$$\left.\begin{aligned} \mathbf{p} &= (1 + \tfrac{1}{2}\lambda k^2 t)\,\mathbf{p}_0 e^{-\frac{1}{2}\lambda k^2 t}, \\ \mathbf{q} &= \tfrac{1}{2}i\lambda k^2 t\mathbf{p}_0 e^{-\frac{1}{2}\lambda k^2 t}. \end{aligned}\right\} \tag{2.15}$$

We shall require an upper bound on the solution in $\mathscr{D}_3$, related to (2.15). For $2^{-\frac{1}{2}} < |\zeta| < 1$, the solution (2.10) is valid; hence, since

$$e^{-(\beta_1-\beta_2)t} = e^{-(\lambda k^2\cos\xi)t} \geqslant 1 - (\lambda k^2\cos\xi)\,t,$$

it follows that

$$\begin{aligned} |\mathbf{p}| &\leqslant \tfrac{1}{2}\tan\xi\, e^{-\beta_2 t}[\cot\tfrac{1}{2}\xi - \tan\tfrac{1}{2}\xi(1 - \lambda k^2 t\cos\xi)]\,|\mathbf{p}_0| \\ &\leqslant e^{-\alpha\lambda k^2 t}[1 + \lambda k^2 t]\,|\mathbf{p}_0|, \end{aligned}$$

where $\alpha = \sin^2\frac{1}{8}\pi$. For $1 < |\zeta| < 2^{\frac{1}{2}}$, the solution (2.14) is valid; hence, since $|\cos\omega t| \leqslant 1$, $|\sin\omega t| \leqslant |\omega t|$,

$$|\mathbf{p}| \leqslant (1 + \tfrac{1}{2}\lambda k^2 t)\,e^{-\frac{1}{2}\lambda k^2 t}|\mathbf{p}_0|. \tag{2.16}$$

Hence

$$|\mathbf{p}| \leqslant e^{-\alpha\lambda k^2 t}(1 + \lambda k^2 t)\,|\mathbf{p}_0| \tag{2.17}$$

throughout $\mathscr{D}_3$ (with equality only if $k^2 t = 0$).

## 3. Suppression of homogeneous turbulence

*(a) Initial conditions*

For homogeneous turbulence, the Fourier–Stieltjes representation

$$\mathbf{u} = \int e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{P}(\mathbf{k}, t), \quad \mathbf{h}(\mathbf{x}, t) = \int e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{Q}(\mathbf{k}, t), \tag{3.1}$$

is appropriate (Batchelor 1953, §2.5). The analysis of §2 still applies, with the replacements

$$\mathbf{p}(\mathbf{k}, t) \rightarrow d\mathbf{P}(\mathbf{k}, t), \quad \mathbf{q}(\mathbf{k}, t) \rightarrow d\mathbf{Q}(\mathbf{k}, t). \tag{3.2}$$

The spectrum tensor of the velocity field is defined by

$$\Phi_{ij}(\mathbf{k}, t) = (2\pi)^{-3} \int \overline{u_i(\mathbf{x}, t)\, u_j(\mathbf{x}+\mathbf{r}, t)}\, e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}, \tag{3.3}$$

and it is related to $d\mathbf{P}(\mathbf{k}, t)$ (Batchelor 1953) by

$$\Phi_{ij}(\mathbf{k}, t) = \lim_{d\mathbf{k}\to 0} \frac{\overline{dP_i^*(\mathbf{k}, t)\, dP_j(\mathbf{k}, t)}}{dk_1\, dk_2\, dk_3}, \tag{3.4}$$

where the bar denotes an ensemble average (in view of the spatial homogeneity, the bar in (3.3) may equally be regarded as a space average). The inverse of (3.4) is

$$\overline{u_i(\mathbf{x}, t)\, u_j(\mathbf{x}+\mathbf{r}, t)} = \int \Phi_{ij}(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}, \tag{3.5}$$

so that in particular

$$K(t) \equiv \tfrac{1}{2}\overline{\mathbf{u}^2} = \frac{1}{2} \int \Phi_{ii}(\mathbf{k}, t)\, d\mathbf{k}. \tag{3.6}$$

Let $\Psi_{ij}(\mathbf{k}, t)$ be the magnetic spectrum tensor defined analogously in terms of $\mathbf{h}$ and satisfying

$$\Psi_{ij}(\mathbf{k}, t) = \lim_{d\mathbf{k}\to 0} \frac{\overline{dQ_i^*(\mathbf{k}, t)\, dQ_j(\mathbf{k}, t)}}{dk_1\, dk_2\, dk_3}. \tag{3.7}$$

An 'interaction tensor' (see Lehnert 1955) may also be defined, but its properties will not be investigated here.

With $\mathbf{h}(\mathbf{x}, 0) \equiv 0$, and so $d\mathbf{Q}(\mathbf{k}, 0) \equiv 0$, it is clear that the results of §2 and the relations (3.4) and (3.7) permit us immediately to write down expressions for $\Phi_{ij}(\mathbf{k}, t)$ and $\Psi_{ij}(\mathbf{k}, t)$ in terms of $\Phi_{ij}(\mathbf{k}, 0)$. We shall be particularly interested in evaluating integral expressions of the type (3.6), and, to this end, the form of $\Phi_{ij}(\mathbf{k}, 0)$ must be specified. Let us suppose that at $t = 0$ the turbulence is isotropic, so that (Batchelor 1953, §3.4)

$$\Phi_{ij}(\mathbf{k}, 0) = \frac{E_0(k)}{4\pi k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{3.8}$$

where $E_0(k)$ is the initial spectrum function satisfying

$$\tfrac{1}{2} u_0^2 \equiv K(0) = \int_0^\infty E_0(k)\, dk. \tag{3.9}$$

We shall suppose that $E_0(k)$ is characterized by a single wave-number $k_0$, so that, on dimensional grounds,

$$E_0(k) = \tfrac{1}{2}u_0^2 k_0^{-1} f(k/k_0), \tag{3.10†}$$

where

$$\int_0^\infty f(\kappa)\,d\kappa = 1. \tag{3.11}$$

Two specific choices of the function $f(\kappa)$ will be considered

$$\text{Prototype A:}\quad f(\kappa) = \frac{C\kappa^4}{1+\kappa^{\frac{17}{3}}}; \tag{3.12}$$

$$\text{Prototype B:}\quad f(\kappa) = C\kappa^4 e^{-\kappa}; \tag{3.13}$$

where, in both cases, $C$ is a constant of order unity determined by (3.11). In both cases

$$E_0(k) \sim \tfrac{1}{2}Cu_0^2 k_0^{-5} k^4 \quad \text{as} \quad k \to 0, \tag{3.14}$$

a behaviour that is known to be dynamically self-preserving in the absence of magnetic effects (i.e. for $t < 0$ here) (Batchelor & Proudman 1956).‡ In case A,

$$E_0(k) \sim \tfrac{1}{2}Cu_0^2 k_0^{\frac{2}{3}} k^{-\frac{5}{3}} \quad (k \gg k_0), \tag{3.15}$$

the behaviour predicted by the Kolmogorov theory for turbulence which has had time (prior to the instant $t = 0$) to attain a state of 'small-scale statistical equilibrium'. The type B spectrum is probably a better representation of turbulence that is created explosively at time $t = 0$ and that does not have time to develop a 'Kolmogorov tail' before the influence of the magnetic field is experienced. Both prototype spectra have maxima at a wave-number of order $k_0$. The length scale of the energy-containing eddies $l_0$, used in § 1, may now be identified with the scale $k_0^{-1}$.

There are now two distinct times which characterize the suppression problem, viz. $t_d$ and

$$t_\lambda = (\lambda k_0^2)^{-1}. \tag{3.16}$$

The ratio of these is

$$\frac{t_\lambda}{t_d} = \left(\frac{h_0}{\lambda k_0}\right)^2 = \left(\frac{k_d}{k_0}\right)^2 = S^2 = NR_m, \tag{3.17}$$

where $S$ is the Lundquist number, already introduced in § 1.

If $S \ll 1$, then $k_d \ll k_0$, and the bulk of the initial energy is contained in $\mathscr{D}_1$ (figure 3*a*), indeed in that part of $\mathscr{D}_1$ where the approximation $|\zeta| \ll 1$ was made. More precisely,

$$K_1(0) = \frac{1}{2}\int_{\mathscr{D}_1} \Phi_{ii}(\mathbf{k}, 0)\,d\mathbf{k} \approx K(0), \tag{3.18}$$

$$K_2(0) = \frac{1}{2}\int_{\mathscr{D}_2} \Phi_{ii}(\mathbf{k}, 0)\,d\mathbf{k} = O(S^5)K(0), \tag{3.19}$$

using (3.14), valid in $\mathscr{D}_2$ in this case.

† The Reynolds number may occur implicitly in (3.10); however, as explained in § 1 an inviscid analysis is appropriate, so that we lose nothing by considering only forms of $E_0(k)$ appropriate to the limit $R \to \infty$.

‡ It has recently been shown (Saffman 1967) that the behaviour $E_0(k) = O(k^2)$ as $k \to 0$ is also a dynamically self-preserving possibility. The analysis that follows can easily be adapted to an initial spectrum which behaves in this way.

If $S \gg 1$, then $k_d \gg k_0$, and the bulk of the initial energy is contained in that part of $\mathscr{D}_2$ for which $|\zeta| \gg 1$ (figure 3*b*). More precisely

$$K_2(0) \approx K(0), \tag{3.20}$$

$$\begin{aligned} K_1(0) &= \int_{\mathscr{D}_1} E_0(k)\, dk\, d\mu \quad (\mu = \cos\theta) \\ &= \int_0^{2^{\frac{3}{2}}k_d} \frac{k}{2^{\frac{3}{2}}k_d} E_0(k)\, dk + \int_{2^{\frac{3}{2}}k_d}^{\infty} E_0(k)\, dk \\ &= O\left(\frac{k_0^2}{k_d} E_0(k_0)\right) + O(k_d E_0(k_d)). \end{aligned} \tag{3.21}$$

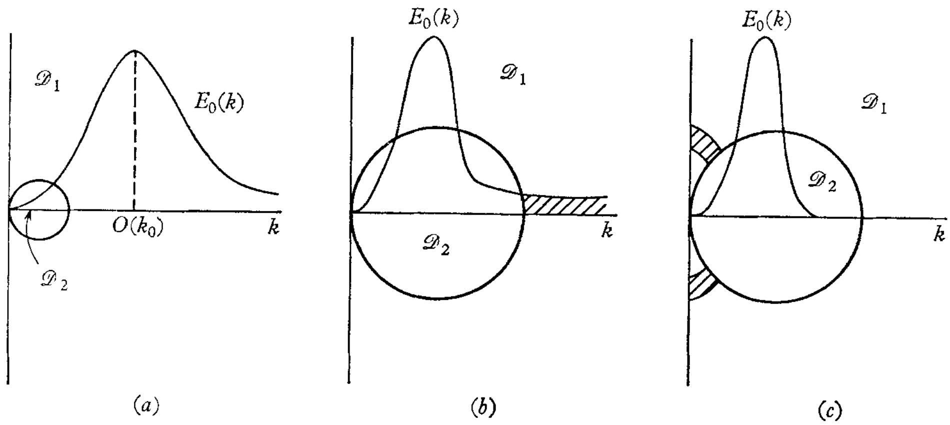



FIGURE 3. The initial energy distribution relative to the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ (the domain $\mathscr{D}_3$ is not shown). The distribution is spherically symmetric, and only its variation with wave-number magnitude is indicated. Case (*a*): $S \ll 1$; case (*b*): $S \gg 1$, type A spectrum; case (*c*): $S \gg 1$, type B spectrum. In cases (*b*) and (*c*) the dominant contribution to the initial energy density comes from the shaded regions.

Here a distinction has to be made between the type A and the type B initial spectra. For a type A spectrum, the second term of (3.21) is dominant and gives

$$K_1(0) = O\left[k_d \left(\frac{k_0}{k_d}\right)^{\frac{5}{3}} E_0(k_0)\right] = O(S^{-\frac{2}{3}})K(0). \tag{3.22}$$

For a type B spectrum, the first term of (3.21) is dominant and gives

$$K_1(0) = O(S^{-1})K(0). \tag{3.23}$$

In this case we shall seek to describe the decay of energy in the time ranges $t_d \ll t \ll t_\lambda$ and $t_\lambda \ll t \ll t_0$. Note that $t_\lambda/t_0 = R_m \ll 1$, by supposition.

### (*b*) *Kinetic energy decay in* $\mathscr{D}_1$

From the solution (2.10) valid in $\mathscr{D}_1$, the replacement (3.2) and the relation (3.4), we have immediately the contribution to $K(t)$ from $\mathscr{D}_1$

$$K_1(t) = \int_{\mathscr{D}_1} \tfrac{1}{4}\tan^2\xi (e^{-\beta_2 t}\cot\tfrac{1}{2}\xi - e^{-\beta_1 t}\tan\tfrac{1}{2}\xi)^2\, \Phi_{ii}(\mathbf{k}, 0)\, d\mathbf{k}. \tag{3.24}$$

Since $|\cot \frac{1}{2}\xi| > |\tan \frac{1}{2}\xi|$ and $\beta_2 < \beta_1$ throughout $\mathscr{D}_1$, the dominant contribution to $K_1(t)$ is given by†

$$K_1(t) \sim \int_{\mathscr{D}_1} \tfrac{1}{4} e^{-2\beta_2 t} \tan^2 \xi \cot^2 \tfrac{1}{2}\xi \, \Phi_{ii}(\mathbf{k}, 0) \, d\mathbf{k}. \tag{3.25}$$

Let $(k, \theta, \chi)$ be spherical polar co-ordinates in wave-number space, and let $\mu = \cos\theta$. Then $d\mathbf{k} = k^2 dk \, d\mu \, d\chi$. The $\chi$-integration in (3.25) is trivial and gives a factor $2\pi$. We then change variables from $(k, \mu)$ to $(k, \xi)$, and, noting that

$$\frac{\partial(k, \mu)}{\partial(k, \xi)} = \frac{\partial \mu}{\partial \xi} = \frac{k}{2k_d} \cos \xi,$$

and that

$$\Phi_{ii}(\mathbf{k}, 0) = E_0(k)/2\pi k^2, \tag{3.26}$$

(3.25) becomes

$$K_1(t) \sim 2 \int_0^\infty \frac{k}{2k_d} \, dk \, E_0(k) \int_0^{\frac{1}{2}\pi} \cos\xi \, d\xi \, \tfrac{1}{4} \tan^2 \xi \cot^2 \tfrac{1}{2}\xi \exp(-2\lambda k^2 \sin^2 \tfrac{1}{2}\xi t). \tag{3.27}$$

For $t \gg t_d$, the dominant contribution to the $\xi$-integral comes from the neighbourhood of $\xi = 0$, so that

$$\begin{aligned} K_1(t) &\sim \int_0^\infty dk \frac{k}{k_d} E_0(k) \int_0^{\frac{1}{2}\pi} \exp(-\tfrac{1}{2}\lambda k^2 \xi^2 t) \, d\xi \\ &= \left(\frac{\pi t_d}{2t}\right)^{\frac{1}{2}} \int_0^\infty E_0(k) \operatorname{erf}\left[\frac{\pi}{4}\left(\frac{\lambda t}{2}\right)^{\frac{1}{2}} k\right] dk, \end{aligned} \tag{3.28}$$

where

$$\operatorname{erf} \eta = \frac{2}{\pi^{\frac{1}{2}}} \int_0^\eta \exp(-\eta_1^2) \, d\eta_1.$$

Now $E_0(k)$ has a maximum at $k = O(k_0)$, so that, provided

$$t \gg (\lambda k_0^2)^{-1} = t_\lambda,$$

the error function in (3.28) is unity over the range of $k$ which contributes significantly to the integral. Hence for $t \gg \max(t_d, t_\lambda)$,

$$K_1(t) \sim (\tfrac{1}{2}\pi)^{\frac{1}{2}} K(0) (t_d/t)^{\frac{1}{2}}. \tag{3.29}$$

Thus, when $S \ll 1$, this result is valid when $t \gg t_d$, and, when $S \gg 1$, it is valid when $t \gg t_\lambda$.

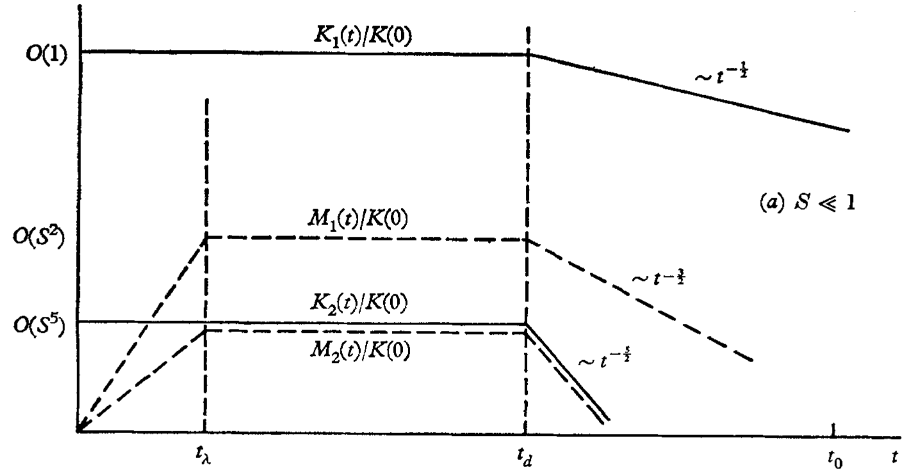
If $S \gg 1$, and $t_d \ll t \ll t_\lambda$, then

$$\operatorname{erf}\left[\frac{\pi}{4}\left(\frac{\lambda t}{2}\right)^{\frac{1}{2}} k\right] \approx \frac{\pi^{\frac{1}{2}}}{2}\left(\frac{\lambda t}{2}\right)^{\frac{1}{2}} k \quad \text{for} \quad k \ll (\lambda t)^{-\frac{1}{2}}. \tag{3.30}$$

The contribution to the integral (3.28) from the range $[(\lambda t)^{-\frac{1}{2}}, \infty]$ is negligible (since $(\lambda t)^{-\frac{1}{2}} \gg k_0$), and (3.28) takes the form

$$K_1(t) \sim \frac{\pi}{2k_d} \int_0^{k_c} k E_0(k) \, dk, \tag{3.31}$$

† It will appear (see (3.29) below) that the retained term here is proportional to $(t_d/t)^{\frac{1}{2}}$, whereas examination of the rejected part of (3.24) shows that it is $O(t_d/t)^{\frac{3}{2}}$, and is always negligible for $t \gg t_d$.



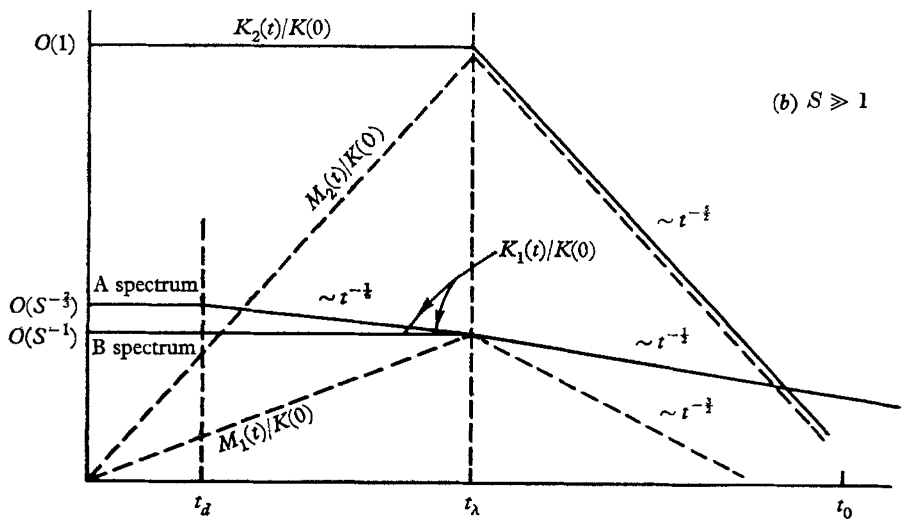




FIGURE 4. The decay of $K_1(t)$, $K_2(t)$, $M_1(t)$, $M_2(t)$, (a) when $S \ll 1$, and (b) when $S \gg 1$. The scale is logarithmic.

where $k_c = O(\lambda t)^{-\frac{1}{2}}$. For a type A spectrum this gives

$$K_1(t)/K_1(0) = O(t_d/t)^{\frac{1}{2}} \quad (t_d \ll t \ll t_\lambda), \tag{3.32}$$

and for a type B spectrum

$$K_1(t)/K_1(0) = O(1) \quad (t_d \ll t \ll t_\lambda). \tag{3.33}$$

These results are summarized in figure 4.

## (c) *Kinetic energy decay in $\mathscr{D}_2$ and $\mathscr{D}_3$*

The contribution to $K(t)$ from $\mathscr{D}_2$ is

$$K_2(t) = \int_{\mathscr{D}_2} \left(\cos \omega t + \frac{\sin \omega t}{\sinh \xi}\right)^2 e^{-\lambda k^2 t} \Phi_{ii}(\mathbf{k}, 0)\, d\mathbf{k}, \tag{3.34}$$

in the notation of §2 (ii). In $\mathscr{D}_2$, $|\sinh\xi| > 1$; hence $K_2(t)$ is of order

$$\int_{\mathscr{D}_2} e^{-\lambda k^2 t}\Phi_{ii}(\mathbf{k},0)\,d\mathbf{k} = \int_0^{k_d} dk \left\{e^{-\lambda k^2 t}E_0(k)\int_{k/k_d}^1 d\mu\right\}$$
$$= \int_0^{k_d} E_0(k)\left(1-\frac{k}{k_d}\right)e^{-\lambda k^2 t}dk. \tag{3.35}$$

If $S \ll 1$, ($k_d \ll k_0$), then we may use the asymptotic form (3.14) in this integral. Hence

$$K_2(t) = O\left\{\tfrac{1}{2}Cu_0^2k_0^{-5}\int_0^{k_d} k^4\left(1-\frac{k}{k_d}\right)e^{-\lambda k^2 t}dk\right\}. \tag{3.36}$$

For $\lambda k_d^2 t \ll 1$, i.e. $t \ll t_d$, $K_2(t)/K_2(0) = O(1)$. For $t \gg t_d$,

$$K_2(t) \sim \tfrac{3}{16}Cu_0^2k_0^{-5}(\lambda t)^{-\frac{5}{2}} = \tfrac{3}{8}CK(0)\,S^5(t_d/t)^{\frac{5}{2}}. \tag{3.37}$$

If $S \gg 1$, ($k_d \gg k_0$), then similarly

$$\left.\begin{aligned} K_2(t)/K_2(0) &= O(1) \qquad && (t \ll t_\lambda),\\ &\sim \tfrac{3}{8}C(t_\lambda/t)^{\frac{5}{2}} && (t \gg t_\lambda).\end{aligned}\right\} \tag{3.38}$$

The behaviour of $K_2(t)$ is also indicated in figure 4.

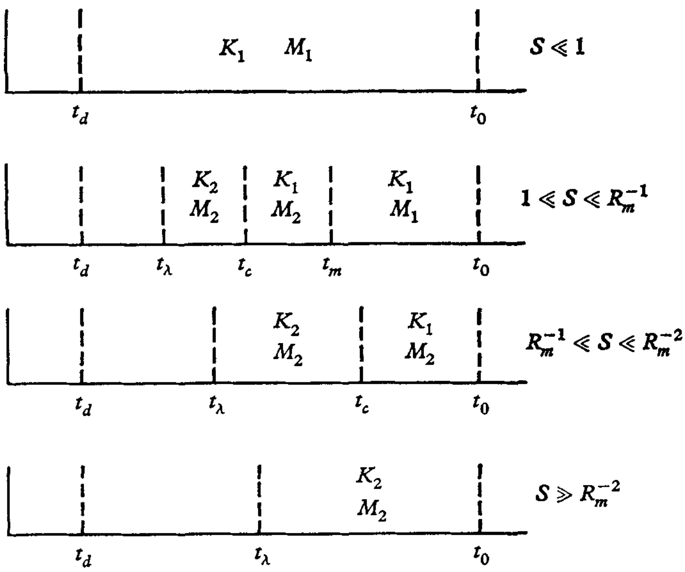



FIGURE 5. An indication of the dominant contributions to kinetic and magnetic energy densities at different stages of the suppression process.

The contribution to $K(t)$ from $\mathscr{D}_3$ satisfies

$$|K_3(t)| < \int_{\mathscr{D}_3}(1+\lambda k^2 t)^2 e^{-2\alpha\lambda k^2 t}\Phi_{ii}(\mathbf{k},0)\,d\mathbf{k}, \tag{3.39}$$

using the inequality (2.17). Using the variables $(\zeta, k)$ this becomes

$$|K_3(t)| < 2\int_{2^{-\frac{1}{2}}}^{2^{\frac{1}{2}}} d\zeta \int_0^{2^{\frac{1}{2}}k_d}(1+\lambda k^2 t)^2 e^{-2\alpha\lambda k^2 t}E_0(k)\frac{k}{2k_d}\,dk.$$

Hence a repetition of the analysis used above for $K_2(t)$ shows that

$$|K_3(t)| \leqslant O[S^5(t_d/t)^3]K(0), \tag{3.40}$$

for $t \gg t_d$ if $S \ll 1$, and for $t \gg t_\lambda$ if $S \gg 1$. Hence $K_3(t)$ is negligible at all times. The situation $S = O(1)$ would pose special difficulties and is not considered here. Since the contribution to the energy from $\mathscr{D}_3$ is negligible (and the same applies to the magnetic energy) it is legitimate to absorb it in $\mathscr{D}_1$ and $\mathscr{D}_2$, which are accordingly henceforth defined by

$$\mathscr{D}_1: |\zeta| < 1; \quad \mathscr{D}_2: |\zeta| > 1. \tag{3.41}$$

*(d) Summary: the decay of $K(t)$*

There are three possibilities that must be distinguished when the contributions $K_1(t)$ and $K_2(t)$ are superposed.

(i) If $S \ll 1$, then $K_2(t)$ is negligible for all $t$, and the asymptotic behaviour is given by equation (3.29). The only significant contribution to the energy comes from the neighbourhood of $\xi = 0$, i.e. $\theta = \frac{1}{2}\pi$; i.e. the turbulence tends to become independent of the co-ordinate $z$ parallel to $\mathbf{h}_0$.

(ii) If $1 \ll S \ll R_m^{-2}$, then $K_2(0) \gg K_1(0)$, but $K_2(t)$ decreases more rapidly than $K_1(t)$ for $t \gg t_\lambda$, and the two contributions become of the same order of magnitude after a time of order $t_c$ where (by comparing (3.29) and (3.38))

$$t_c = S^{\frac{1}{2}}t_\lambda = S^{\frac{3}{2}}t_d = S^{\frac{1}{2}}R_m t_0. \tag{3.42}$$

For $t_\lambda \ll t \ll t_c$, $K(t) \approx K_2(t)$ and is given by (3.38), while, for $t_c \ll t \ll t_0$, $K(t) \approx K_1(t)$ and is given by (3.29). Again for $t \gg t_c$, the turbulence is nearly $z$-independent.†

(iii) If $S \gg R_m^{-2}$, then $K_2(t) \gg K_1(t)$ throughout the suppression process. When $t = O(t_0)$, the turbulence is still nearly isotropic (since in $\mathscr{D}_2$ there is no strong tendency to anisotropy—see §4 below) and has the character of a random superposition of slowly decaying Alfvén waves.

*(e) The ultimate partition of kinetic energy parallel and perpendicular to the applied field*

The initial isotropy implies that at $t = 0$,

$$\overline{u^2} = \overline{v^2} = \overline{w^2} = \tfrac{1}{2}(\overline{u^2}+\overline{v^2}), \tag{3.43}$$

where $\mathbf{u} = (u, v, w)$. If $S \gg R_m^{-2}$, then this isotropy persists throughout the suppression process. If $S \ll R_m^{-2}$, then ultimately the turbulence may be described as 'nearly two-dimensional', in the sense that all correlations vary slowly in the $z$-direction parallel to $\mathbf{h}_0$ compared with their variation perpendicular to $\mathbf{h}_0$. This does not imply, however, that $\overline{w^2} \to 0$; indeed, for $t \gg \max(t_d, t_c)$,

$$\left.\begin{aligned} \overline{w^2} &\sim \int_{\mathscr{D}_1} \Phi_{33}(\mathbf{k}, t)\, d\mathbf{k}, \\ \overline{u^2}+\overline{v^2} &\sim \int_{\mathscr{D}_1} [\Phi_{11}(\mathbf{k}, t) + \Phi_{22}(\mathbf{k}, t)]\, d\mathbf{k}. \end{aligned}\right\} \tag{3.44}$$

† If non-linear effects become important at the earlier time $t_n = N^\gamma t_0$ then the change-over will occur only if $t_c \ll t_n$, i.e. if $S \ll R_m^{-2\gamma/(5-4\gamma)}$ (see footnote on p. 572).

Now from (3.8),

$$\left.\begin{aligned}\Phi_{33}(\mathbf{k},0) &= \frac{E_0(k)}{4\pi k^2}(1-\cos^2\theta),\\ \Phi_{11}(\mathbf{k},0)+\Phi_{22}(\mathbf{k},0) &= \frac{E_0(k)}{4\pi k^2}(1+\cos^2\theta).\end{aligned}\right\} \tag{3.45}$$

Since the dominant contribution to the integrals (3.44) comes from the neighbourhood of $\theta = \frac{1}{2}\pi$, the contribution from the $\cos^2\theta$ terms is negligible, and so ultimately

$$\overline{w^2} \approx \overline{u^2}+\overline{v^2}, \tag{3.46}$$

in contrast to (3.43). The energy is therefore, to some extent, channelled during suppression into the component parallel to the field.

### (*f*) *Magnetic energy development and decay*

The magnetic energy may be treated in the same way as the kinetic energy. It will be enough to indicate here the essential differences and conclusions. The magnetic energy rises from zero (by virtue of the special assumption $\mathbf{h}(\mathbf{x},0) = 0$) to its maximum value in a time of order $t_\lambda$. For $t \gg t_\lambda$

$$M_2(t) \approx K_2(t) = O[K_2(0)\,(t_\lambda/t)^{\frac{1}{2}}]; \tag{3.47}$$

i.e. in $\mathscr{D}_2$ there is approximate equipartition of energy; this follows essentially from the behaviour of the Fourier components (2.15) for $|\zeta| \gg 1$. In $\mathscr{D}_1$ the main contribution to $M_1(t)$ again comes from the neighbourhood of $\theta = \frac{1}{2}\pi$, where, as observed after equation (2.13), the magnetic energy in each Fourier component is a factor $(k_d/k_0)^2\cos^2\theta$ smaller than the kinetic energy. The integral over $\mathscr{D}_1$ of $\Psi_{ii}(\mathbf{k},t)$ for $t \gg t_\lambda$ is of order

$$M_1(t) \approx \int_{k_d}^{\infty}\left(\frac{k_d}{k}\right)^2 E_0(k)\,dk\int_0^1 \mu^2 e^{-2\mu^2 t/t_d}\,d\mu, \tag{3.48}$$

and for $t \gg \max(t_d, t_\lambda)$ this gives

$$M_1(t) = O[S^2K(0)(t_d/t)^{\frac{3}{2}}], \tag{3.49}$$

which may be compared with (3.29). Note that

$$M_1(t)/K_1(t) = O[S^2(t_d/t)] = O(t_\lambda/t). \tag{3.50}$$

If $S \ll 1$, then $M(t) \approx M_1(t)$ at all times. If $S \gg 1$, then, as for the kinetic energy, $M_2(t) \gg M_1(t)$ up to a time of order $t_m$, where, by comparing (3.47) and (3.49),

$$t_m = S^3 t_d. \tag{3.51}$$

If $t_m \ll t_0$, i.e. if $1 \ll S \ll R_m^{-1}$, then

$$\left.\begin{aligned}M(t) &\approx M_2(t) \quad (t_\lambda \ll t \ll t_m),\\ M(t) &\approx M_1(t) \quad (t_m \ll t \ll t_0).\end{aligned}\right\} \tag{3.52}$$

If $S \gg R_m^{-1}$, however, then $t_m \gg t_0$, so that $M(t) \approx M_2(t)$ throughout the suppression process.† The various possibilities are summarized in figure 5. Note that

† In this case $t_m \ll t_n \Leftrightarrow S \ll R_m^{-\gamma/(3-2\gamma)}$; cf. footnote on p. 584.

if $R_m^{-1} \ll S \ll R_m^{-2}$, then, when $t = O(t_0)$, the dominant contribution to $K(t)$ is $K_1(t)$ while the dominant contribution to $M(t)$ is still $M_2(t)$; in this case the ultimate velocity field is nearly $z$-independent, while the ultimate magnetic field is still nearly isotropic (see equation (4.2) *et seq.*)

## 4. Discussion

Let us now summarize and elucidate the principal results of the preceding section. For $0 < t/t_d \ll N$ the response of the turbulence to the applied field is linear. The nature of the response is largely determined by the initial spectral distribution of kinetic energy relative to the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ of $\mathbf{k}$-space. If $S \ll 1$, the bulk of the initial energy is in $\mathscr{D}_1$, and it decays for $t \gg t_d$ as $(t_d/t)^{\frac{1}{2}}$. The dominant contribution to the energy for $t \gg t_d$ comes from the region $|\cos\theta| = |\mu| \ll 1$ of low ohmic dissipation, where the equation for the spectrum tensor takes the approximate form

$$\frac{\partial}{\partial t}\Phi_{ij}(\mathbf{k},t) = -\frac{2}{t_d}\mu^2\Phi_{ij}(\mathbf{k},t). \tag{4.1}$$

The motion tends to become independent of the co-ordinate in the direction of $\mathbf{h}_0$. The magnetic energy in $\mathscr{D}_1$ is small compared with the kinetic energy and decays as $(t_d/t)^{\frac{3}{2}}$.

If $S \gg 1$, the bulk of the initial energy is in $\mathscr{D}_2$, the domain of slowly decaying Alfvén waves. Both kinetic and magnetic energy decay as $(t_\lambda/t)^{\frac{5}{2}}$ for $t \gg t_\lambda$. The dominant contribution to this energy ultimately comes from the neighbourhood of $\mathbf{k} = 0$ in $\mathbf{k}$-space, actually from Alfvén waves whose wavelength is of order $(\lambda t)^{\frac{1}{2}}$. (If the turbulence is not exactly homogeneous, but is confined to a region of finite span $L_0$, the description breaks down after a time of order $\lambda^{-1}L_0^2$ and for $t \gg \lambda^{-1}L_0^2$ the $\mathscr{D}_2$ energy contributions decay exponentially.) If sufficient time is available before non-linear effects became significant, the $\mathscr{D}_1$ component, no matter how small initially, will ultimately dominate ($K_1$ dominating $K_2$ somewhat earlier than $M_1$ dominates $M_2$). The spectrum tensor in the region $|\zeta| \gg 1$ where the $\mathscr{D}_2$ energy is concentrated is, from the solution (2.14),

$$\Phi_{ij}(\mathbf{k},t) = \Phi_{ij}(\mathbf{k},0)\,e^{-\lambda k^2 t}\cos^2(\mathbf{h}_0\,.\,\mathbf{k})\,t. \tag{4.2}$$

The rapid variation with $\theta$ is physically unimportant and, for $t \gg (h_0 k)^{-1}$, the 'smoothed' isotropic decay,

$$\Phi_{ij}(\mathbf{k},t) \approx \tfrac{1}{2}\Phi_{ij}(\mathbf{k},0)\,e^{-\lambda k\ t}, \tag{4.3}$$

is a reasonable simplification of (4.2). Similarly, the magnetic energy tensor takes the same approximate form.

The physical reason for the decay being isotropic in this case is broadly that the decay factor for each Fourier component is $e^{-\lambda k^2 t}$, independent of $\theta$; the waves travel with a wave velocity $h_0 k\cos\theta$, and this leads to the factor $\cos^2(\mathbf{h}_0\,.\,\mathbf{k})\,t$ in (4.2); the magnetic spectrum tensor involves the factor $\sin^2(\mathbf{h}_0\,.\,\mathbf{k})\,t$, the magnetic perturbation being exactly out of phase with the velocity in each Fourier component (for small enough $k$).

Some numerical computations that are relevant to the present study have been made by Deissler (1963). The computations were made for several values of the ratio $\nu/\lambda$, and only the results for $\nu/\lambda \ll 1$ need be considered here. The main conclusions for this case were (i) that the magnetic energy was negligibly small compared with the kinetic energy for as long as the computations were continued; (ii) that $3\overline{w^2}/\overline{\mathbf{u}^2}$ increases from 1 to $\frac{3}{2}$ during the process of suppression. Both conclusions are borne out by the analysis of this paper in the case $S \ll 1$ (see §3 (*e*) and (*f*)). Perusal of Deissler's paper shows that his computations are in fact relevant only to this case; the asymptotic form (3.14) is adopted as an initial condition for all $k$ and this is legitimate only if the bulk of the initial energy is in $\mathscr{D}_1$, i.e. only if $S \ll 1$.

When $t = O(t_0)$ and later, non-linear terms in the equation of motion will tend to redistribute energy in wave-number space, and thus lead indirectly to a change in the laws of energy decay. The prevailing tendency in ordinary turbulence is for non-linear terms to transfer energy towards the viscous sink at high wave-numbers. In the problem under consideration, in addition to this effect, we should expect a transfer of energy from the neighbourhood of $\mu = 0$ (where there is no ohmic dissipation) towards the region $\mu = O(1)$, where ohmic dissipation is strong. If $S$ is small this will undoubtedly accelerate the decay of energy in $\mathscr{D}_1$; i.e. $K(t)$ will decrease more rapidly than $(t_d/t)^{\frac{1}{2}}$. It is clear from a consideration of (4.1) that the non-linear term will first become significant in the important neighbourhood of $\mu = 0$, where the only other term contributing to spectral change is small. Relative to the magnetic force, the inertia force is most important in this region, and its influence is therefore likely to be felt at an earlier stage than the original crude estimate would suggest (see footnote on p. 2).

If $S$ is sufficiently large, non-linear effects may be less important. The Alfvén waves in $\mathscr{D}_2$ are of small amplitude in the sense $(\overline{\mathbf{u}^2})^{\frac{1}{2}}, (\overline{\mathbf{h}^2})^{\frac{1}{2}} \ll h_0$ provided $S \gg R$ (from a consideration of the definition of these two numbers), and interaction of decaying Alfvén waves may do no more than slowly alter the spectral shape as described by the linear theory (equation (4.3)). However, there may be a significant transfer of energy from $\mathscr{D}_2$ to $\mathscr{D}_1$. In the absence of a detailed model for the process of non-linear transfer, it is impossible to say in which domain of $\mathbf{k}$-space the dominant contribution to the energy ultimately resides.

## Appendix. Relation with previous work on the final period of decay

As mentioned in the introduction, the linearized MHD equations were studied by Lehnert (1955) in the context of the final period of decay, that is, under circumstances wherein the velocity and magnetic fluctuations are sufficiently weak for non-linear effects to be permanently negligible; sufficient conditions for this to be the case are

$$R \ll 1, \quad R_m \ll 1, \tag{A 1}$$

$R$ and $R_m$ being based on the r.m.s. velocity and on $l_0 = k_0^{-1}$, the length scale of the energy-containing eddies at some initial instant marking the 'beginning of the final period'. The viscous term $\nu\nabla^2\mathbf{u}$ must then be included in the equation of motion (1.7); otherwise the equations studied in this paper are unaltered.

Several authors have continued the study of Lehnert's equations and it may be as well to review briefly their conclusions, and to set them in perspective with the present paper. The question of the asymptotic law of decay appears to have been first examined by Alexandrou (1963), who demonstrated that, when $\lambda = \nu$, the kinetic and magnetic energy densities were asymptotically (i.e. for $t \to \infty$) equal, and proportional to $t^{-\frac{5}{2}}$ as for non-magnetic turbulence (Batchelor 1953, §5.4). The result depends critically on the behaviour of the spectrum tensors $\Phi_{ij}(\mathbf{k}, 0)$ and $\Psi_{ij}(\mathbf{k}, 0)$ near $\mathbf{k} = 0$, since essentially it is the Fourier components of largest scale which survive longest. The nature of the spectral tensors in this neighbourhood is determined by non-linear interactions during the complete history of the turbulence prior to the final period of decay. This aspect of the problem was examined thoroughly by Alexandrou following the assumption and method of Batchelor & Proudman (1956). The essential property of the spectral tensors that he obtained for axisymmetric turbulence was

$$\Phi_{ii}(\mathbf{k}, 0) \sim k^2 f_0(\mu^2), \quad \Psi_{ii}(\mathbf{k}, 0) \sim k^2 g_0(\mu^2) \quad (\mathbf{k} \to 0), \tag{A 2}$$

where $f_0$ and $g_0$ are functions which depend on the previous (i.e. $t < 0$) non-linear history of the turbulence.† It is then integrals of the form

$$\int \Phi_{ii}(\mathbf{k}, 0) \cos^2(h_0 k \mu t)\, e^{-\lambda k^2 t} d\mathbf{k},$$

which asymptotically give contributions proportional to $t^{-\frac{5}{2}}$. Alexandrou argued further that the $t^{-\frac{5}{2}}$ law was likely to be asymptotically valid if $\lambda \neq \nu$ provided $\lambda/\nu = O(1)$.

The decay of turbulence during the final period in the weakly conducting case has been further considered by Lecocq (1962), Eliseev (1965*a*, *b*) and Nihoul (1965, 1966). Lecocq obtained the particular form of Lehnert's spectral decay results appropriate to the situation $\nu \ll \lambda$. Eliseev, and Nihoul (1965), examined the decay of kinetic energy density during the final period, and independently obtained the result

$$K(t) \sim t^{-3} \quad (t/t_d \to \infty). \tag{A 3}$$

In essence, in these treatments, only the contribution to $K(t)$ from the domain $\mathscr{D}_1$ of $\mathbf{k}$-space was considered. However, the contribution from $\mathscr{D}_2$ in fact falls off more slowly than that from $\mathscr{D}_1$, and Nihoul (1966) has now given the modified result

$$K(t) \propto \begin{cases} t^{-3} & (t_d \ll t \ll (\lambda/\nu)^5 t_d), \\ t^{-\frac{5}{2}} & (t \gg (\lambda/\nu)^5 t_d). \end{cases} \tag{A 4}$$

It has not previously been recognized that the decay of $K(t)$ in the final period must depend significantly on the value of the initial Lundquist number $S$ as well as on the ratio $\lambda/\nu$. It will now be shown that Nihoul's result (A 4) is essentially correct if $S \ll (\nu/\lambda)^{\frac{1}{2}} \ll 1$ (see (A 13) below), but that it requires further modification if $S \gg (\nu/\lambda)^{\frac{1}{2}}$.

† In view of Saffman's (1967) work, it is now evident that this is perhaps not the most general possibility.

In order to describe the final period of decay, the analysis of §§ 2–3 requires modification only through the inclusion of the viscous term $\nu\nabla^2\mathbf{u}$ in the equation of motion (1.7). The modification in the analysis of § 2 is trivial. The definition (2.5) must be replaced by

$$k_d = h_0/|\lambda-\nu|, \tag{A 5}$$

and (2.3) becomes

$$\beta_{1,2} = \tfrac{1}{2}(\lambda+\nu)k^2 \pm \tfrac{1}{2}(\lambda-\nu)k^2(1-\zeta^2)^{\frac{1}{2}}. \tag{A 6}$$

If $\lambda = \nu$, then $k_d = \infty$, so that the whole $\mathbf{k}$-space (except the singular plane $\theta = \frac{1}{2}\pi$) lies in $\mathscr{D}_2$. In this case the analysis of § 3 (*c*) again gives

$$K_2(t)/K_2(0) \propto (t_\lambda/t)^{\frac{5}{2}} \quad (t \gg \max(t_\lambda, t_d)), \tag{A 7}$$

consistent with Alexandrou (1963). This result is insensitive to the choice of the functions $f_0$ and $g_0$ appearing in the initial conditions (A 2) apart from a constant of proportionality of order unity; all that is required is the factor $k^2$ in these conditions.

Suppose now that $\lambda \gg \nu$ (results for the case $\lambda \ll \nu$ may be inferred by virtue of the symmetry of the governing equations). Then $k_d \approx h_0/\lambda$ as in § 2, and the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ are as before. There are now three time scales that must be considered:

$$t_d = \lambda/h_0^2, \quad t_\lambda = (\lambda k_0^2)^{-1}, \quad t_\nu = (\nu k_0^2)^{-1}, \tag{A 8}$$

and, by assumption, $t_\nu \gg t_\lambda$.

It is not difficult to see that the decay in $\mathscr{D}_2$ is not materially changed when $\nu$ is non-zero; the amplitude of each Fourier component decays approximately as $e^{-\frac{1}{2}\lambda k^2 t}$, and the decay of $K_2(t)$ is still given by (A 7). However viscous effects do ultimately become important in $\mathscr{D}_1$; in the region $|\zeta| \ll 1$, the roots (A 6) become

$$\beta_1 \approx \lambda k^2, \quad \beta_2 \approx \lambda k_d^2 \cos^2\theta + \nu k^2. \tag{A 9}$$

This leads to

$$K_1(t) \approx \int_{\mathscr{D}_1} \Phi_{ii}(\mathbf{k}, 0) \exp\{-2(\mu^2 t/t_d + \nu k^2 t)\} d\mathbf{k}.$$

Clearly for $t \ll t_\nu$ the viscous factor is unimportant. The factor $\exp(-2t\mu^2/t_d)$ contributes a factor $(t_d/t)^{\frac{1}{2}}$ to $K_1(t)$ for $t \gg \max(t_d, t_\lambda)$ (just as in § 3 (*c*)) and the factor $e^{-2\nu k^2 t}$ contributes a factor $(t_\nu/t)^{\frac{5}{2}}$ to $K_1(t)$ for $t \gg t_\nu$. (Here we consider only a type B initial spectrum, this being the more appropriate for a final period of decay calculation.) Hence if $t_d \ll t_\nu$, then

$$\left.\begin{aligned} K_1(t) &\approx K_1(0)(t_d/t)^{\frac{1}{2}} \quad \text{for} \quad \max(t_d, t_\lambda) \ll t \ll t_\nu, \\ &\approx K_1(0)(t_d/t)^{\frac{1}{2}}(t_\nu/t)^{\frac{5}{2}} \quad \text{for} \quad t \gg t_\nu, \end{aligned}\right\} \tag{A 10}$$

and, if $t_d \gg t_\nu$,

$$\left.\begin{aligned} K_1(t) &\approx K_1(0)(t_\nu/t)^{\frac{5}{2}} \quad \text{for} \quad t_\nu \ll t \ll t_d, \\ &\approx K_1(0)(t_d/t)^{\frac{1}{2}}(t_\nu/t)^{\frac{5}{2}} \quad \text{for} \quad t \gg t_d. \end{aligned}\right\} \tag{A 11}$$

In these results, and in those that follow, constants of order unity are omitted.

When the contributions from $\mathscr{D}_1$ and $\mathscr{D}_2$ are superposed, there are essentially four distinct possibilities summarized in figure 6.

(i) $S \ll (\nu/\lambda)^{\frac{1}{2}} \ll 1$, i.e. $t_\lambda \ll t_\nu \ll t_d$. In this case, $K_1(t)$ dominates until a time

$$t_q = (\lambda/\nu)^5 t_d, \tag{A 12}$$

and

$$\frac{K(t)}{K(0)} \approx \begin{cases} (t_\nu/t)^{\frac{5}{2}} & (t_\nu \ll t \ll t_d), \\ (t_\nu/t)^{\frac{5}{2}}(t_d/t)^{\frac{1}{2}} & (t_d \ll t \ll t_q), \\ S^5(t_d/t)^{\frac{5}{2}} & (t \gg t_q), \end{cases} \tag{A 13}$$

consistent with Nihoul's (1967) result (A 3).

(ii) $(\nu/\lambda)^{\frac{1}{2}} \ll S \ll 1$, i.e. $t_\lambda \ll t_d \ll t_\nu$. In this case $K_1(t)$ again dominates until the time $t_q$; however, (A 13) clearly requires modification

$$\frac{K(t)}{K(0)} \approx \begin{cases} (t_d/t)^{\frac{1}{2}} & (t_d \ll t \ll t_\nu), \\ (t_\nu/t)^{\frac{5}{2}}(t_d/t)^{\frac{1}{2}} & (t_\nu \ll t \ll t_q), \\ S^5(t_d/t)^{\frac{5}{2}} & (t \gg t_q). \end{cases} \tag{A 14}$$

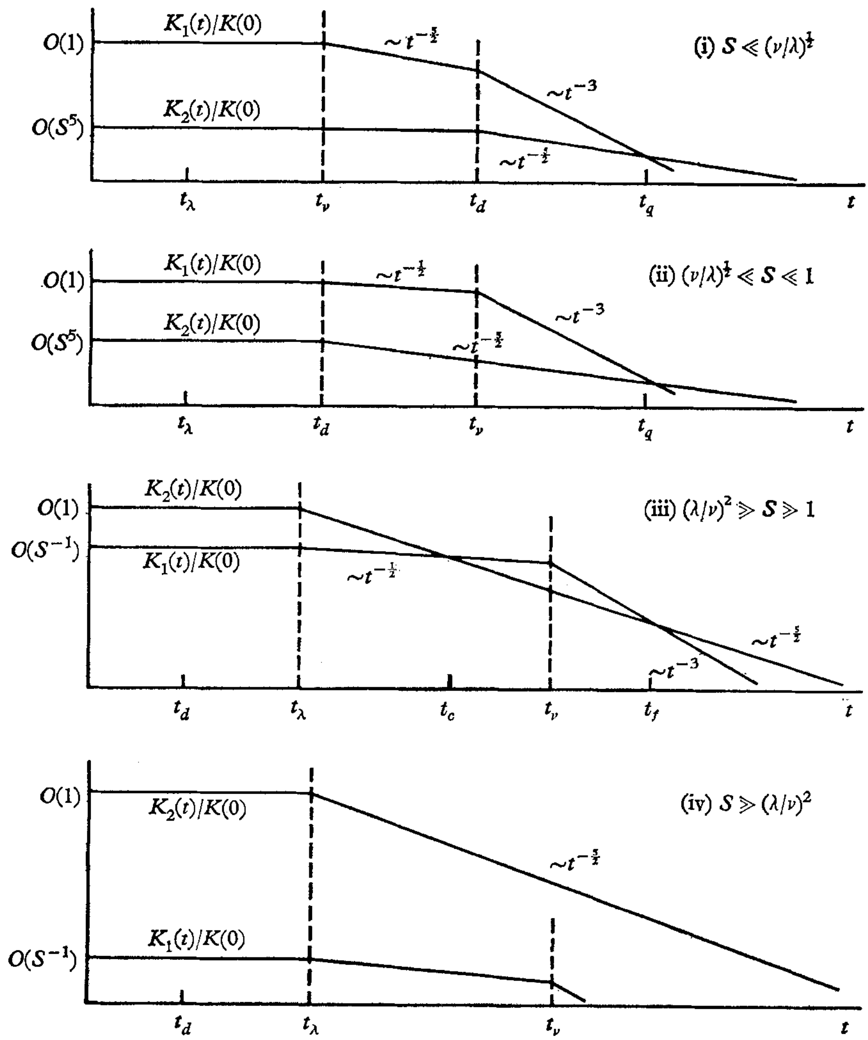



FIGURE 6. Decay of $K_1(t)$ and $K_2(t)$ in the final period for different orders of magnitude of the parameter $S$. The scale is logarithmic.

(iii) $1 \ll S \ll (\lambda/\nu)^2$. In this case $K_2(t)$ dominates for $t \ll t_c$ and for $t \gg t_f$, where

$$t_c = S^{\frac{1}{2}} t_\lambda, \quad t_f = S^{-2}(\lambda/\nu)^4 t_\nu, \tag{A 15}$$

but $K_1(t)$ dominates for $t_c \ll t \ll t_f$. Hence

$$\frac{K(t)}{K(0)} \approx \begin{Bmatrix} (t_\lambda/t)^{\frac{5}{2}} & (t_\lambda \ll t \ll t_c), \\ S^{-1}(t_\lambda/t)^{\frac{1}{2}} & (t_c \ll t \ll t_\nu), \\ S^{-1}(t_\lambda/t)^{\frac{1}{2}}(t_\nu/t)^{\frac{5}{2}} & (t_\nu \ll t \ll t_f), \\ (t_\lambda/t)^{\frac{5}{2}} & (t \gg t_f). \end{Bmatrix} \tag{A 16}$$

(iv) $S \gg (\lambda/\nu)^2 \gg 1$. In this case $K_2(t)$ dominates throughout the final period of decay, and

$$K(t) \approx K(0)\,(t_\lambda/t)^{\frac{5}{2}} \quad (t \gg t_\lambda). \tag{A 17}$$

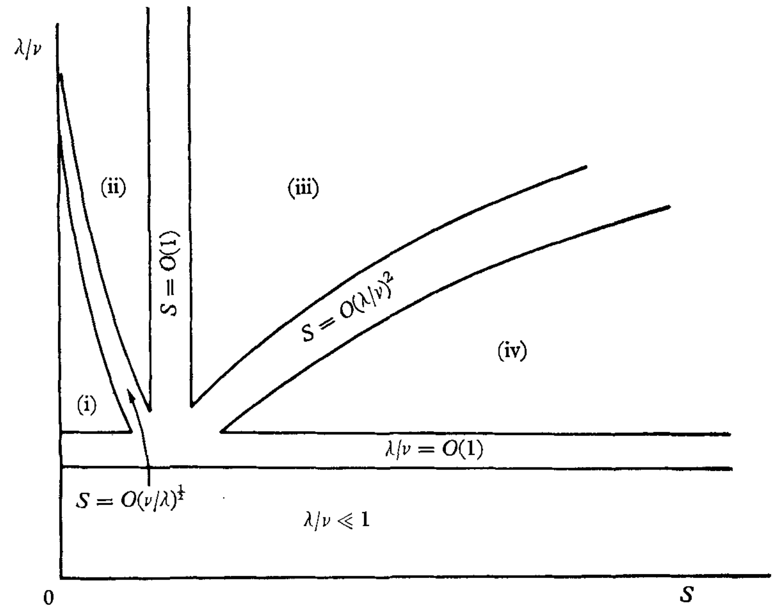



FIGURE 7. Schematic division of the $(S, \lambda/\nu)$-plane into the four regions (for $\lambda/\nu \gg 1$) in which distinct patterns of decay occur. The numbers (i)–(iv) correspond to the numbers used in the text. The region $\lambda/\nu \ll 1$ could be similarly divided.

The reason that the cases (ii), (iii) and (iv) are not embraced by Nihoul's analysis is that, like Deissler, he adopts initial conditions of the form (A 2) and applies them as though they are valid for all $k$; this is legitimate only in the limit $k_0 \to \infty$, i.e. $S \to 0$, so that only the case (i), in which $S$ is smaller than any other small numbers appearing in the analysis, is adequately described.

The regions of validity of the results (A 13)–(A 17) in the plane of the variables $(S, \lambda/\nu)$ is indicated in figure 7. In the limit $\lambda/\nu \to \infty$, for fixed $S$, the regions (ii) and (iii) survive, and $t_\nu \to \infty$. The spectacular variability of the results in the different regions in this trivial type of linear turbulence may serve as a warning of the complexity of behaviour that may be anticipated under fully developed non-linear conditions.

REFERENCES


ALEXANDROU, N. 1963 Ph.D. Thesis, King's College, London.

BATCHELOR, G. K. 1953 *Homogeneous Turbulence.* Cambridge University Press.

BATCHELOR, G. K. & PROUDMAN, I. 1956 *Phil. Trans.* A **248**, 369.

DEISSLER, R. B. 1963 *Phys. Fluids* **6**, 1250.

ELISEEV, B. V. 1965*a* *Soviet Physics, Doklady* **10**, 239.

ELISEEV, B. V. 1965*b* *J. Appl. Math. Mech.* (PMM) **29**, 1133.

GOLITSYN, G. S. 1960 *Soviet Physics, Doklady* **5**, 536.

LECOCQ, P. 1962 *C. R. Acad. Sc. Paris*, **254**, 3633.

LEHNERT, B. 1955 *Quart. Appl. Math.* **12**, 321.

LIEPMANN, H. W. 1952 *Z. Appl. Math. Phys.* **3**, 321.

MOFFATT, H. K. 1962 *Mécanique de la Turbulence*, p. 395. Editions du C.N.R.S. no. 108. Paris.

NESTLERODE, J. A. & LUMLEY, J. L. 1963 *Phys. Fluids* **6**, 1260.

NIHOUL, J. C. J. 1965 *Physica* **31**, 141.

NIHOUL, J. C. J. 1966 *Phys. Fluids* **9**, 2370,

SAFFMAN, P. G. 1967 *J. Fluid Mech.* **27**, 581.